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Petersen, Mark A.; Schaft, Arjan J. van der

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# Nonsquare Spectral Factorization for Nonlinear Control Systems

Mark A. Petersen and Arjan J. van der Schaft

**Abstract**—This paper considers nonsquare spectral factorization of nonlinear input affine state space systems in continuous time. More specifically, we obtain a parametrization of nonsquare spectral factors in terms of invariant Lagrangian submanifolds and associated solutions of Hamilton–Jacobi inequalities. This inequality is a nonlinear analogue of the bounded real lemma and the control algebraic Riccati inequality. By way of an application, we discuss an alternative characterization of minimum and maximum phase spectral factors and introduce the notion of a rigid nonlinear system.

**Index Terms**—Hamilton–Jacobi inequalities, invariant Lagrangian manifolds, nonlinear nonsquare spectral factors.

## I. INTRODUCTION

THE multivariable spectral factorization problem originated several decades ago in such areas of interest as stochastic realization theory (see [21] and [59]), network synthesis (see [1]) and control (see [56]). More recently, applications of this problem to linear and nonlinear systems and control theory (see [13] and [19]), in particular  $H_\infty$ -control [9], [12], [29], [30], [36], [37], [61], chemical process control (see [20] and [69]), geometric control theory (see [28] and the references therein) and stochastic (realization) theory (see [27], [41], and [46]) have continued to evolve. Despite the fact that it is difficult to factor nonlinear systems, nonlinear versions of spectral factorization and the associated concept of inner–outer factorization have been analyzed by using various techniques in [3]–[7], [9], [10], [12], [13], [31], [48], [54], [55], and [64].

Our main aim in this paper is to establish a nonlinear analogue of certain parametrizations of the special class of nonsquare minimal spectral factors that are stable. This we will accomplish in terms of nonlinear analogues of invariant subspaces and related solutions of algebraic Riccati equations. Our investigation of the nonlinear spectral factorization problem outlined above is motivated by its importance in applications in systems and control theory. For instance, we can identify several relationships with the control of mechanical systems (see, for instance, [43] and [64]),  $H_\infty$ -control (see [12]) and chemical process control (see, for instance, [20] and [69]). A further reason for investigating this problem is its connection with stochastic control via

stochastic realization theory; although this connection may not be as well-understood as in the linear case. In particular, in stochastic realization problems any stable spectral factor, square or nonsquare, is interesting to consider (see [40] and [42]). Also, in economics, one could try to explain a multivariable time series in terms of the dynamics of less variables (factors) and white noise disturbances. Multichannel signal transmission is another reason for understanding the nonsquare spectral factorization problem. In practice, this has implications for (amongst other things) mobile phones that have to decode signals from surrounding transmitters that themselves have to be able to transmit signals to other phones.

Next, we discuss the contribution of the current paper in relation to recent literature on the subject of spectral factorization of nonlinear control systems. The first attempts to understand nonlinear, nonsquare, stable, and spectral factors were made in [49] and [54]. In those contributions, we obtain a parametrization of nonsquare spectral factors in terms of an invariant Lagrangian submanifold but make no attempt to understand the related extremal factors. This paper is an extension to the nonsquare case of results determined for nonlinear, minimal, square, and stable spectral factors by Ball and Petersen in [10] and Ball and van der Schaft in [13]. In [10], we extended the notion of the correspondence between minimal square spectral factors of a given spectral density, invariant subspaces of an associated Hamiltonian matrix, solutions of an appropriate algebraic Riccati equation and minimal unitary left divisors of a certain unitary function on the imaginary line to the nonlinear scenario. In particular, we exposed a bijective equivalence between nonlinear, minimal, square, stable spectral factors, invariant Lagrangian submanifolds, solutions of Hamilton–Jacobi equations, and minimal right inner divisors.

Because the current paper treats the situation where the nonlinear, minimal, stable spectral factors may be nonsquare the techniques used are more sophisticated. By contrast to [10], the work in this paper entails studying a slightly different type of Hamilton–Jacobi equation when parametrizing spectral factors in terms of invariant Lagrangian submanifolds. In [13], a nonlinear square state space system is expressed as the cascade connection of an inner (lossless) system and a stable minimum phase (outer) system that is found to be a solution of an associated nonlinear spectral factorization problem. This idea also has an important part to play in the present paper.

## II. NONLINEAR SPECTRAL FACTORIZATION PROBLEM

In this section, we provide a brief description of the class of nonlinear control systems that we analyze. Also, we consider recent spectral factorization results for linear systems that we

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M. A. Petersen is with the Department of Mathematics and Applied Mathematics, North-West University (Potchefstroom Campus), Potchefstroom x6001, South Africa (e-mail: wskmap@puk.ac.za).

A. J. van der Schaft is with the Faculty of Mathematical Sciences, University of Twente, 7500 AE Enschede, The Netherlands (e-mail: twarjan@math.utwente.nl).

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generalize to the nonlinear case. In Section II-C, we formulate the main problem that we will solve in Section III. Finally, we comment on the novel features and the format of this paper.

### A. Description of the Nonlinear Control System

The system that we consider is **input affine**, i.e., its input variables are linear. This assumption ensures that we obtain explicit formulas for our factors that resemble those found in the linear case. We consider a smooth nonlinear input-affine system

$$\Sigma : \begin{cases} \dot{x} = a(x) + b(x)u, & u \in \mathbf{R}^m \\ y = c(x) + d(x)u, & y \in \mathbf{R}^p \end{cases} \quad (1)$$

where we have that  $a : \mathbf{R}^n \rightarrow \mathbf{R}^n$ ,  $b : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times m}$ ,  $c : \mathbf{R}^n \rightarrow \mathbf{R}^p$  and  $d : \mathbf{R}^n \rightarrow \mathbf{R}^{p \times m}$  is a smooth function (at least  $C^1$ ). Here, we assume that (1) is of **full rank**, where we make use of the Nijmeijer–van der Schaft definition of the rank of a nonlinear system given in [44]. We will suppose that  $p \geq m$  and that  $d(x)$  is injective for all  $x$ , where  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$  are local coordinates for the  $n$ -dimensional state–space manifold  $\mathcal{M}$ , with globally asymptotically stable equilibrium  $x_0 = 0$  for  $u = 0$  (so  $a(x_0) = 0$  and  $c(x_0) = 0$ ). From this it follows that  $E(x) := d(x)^T d(x)$  is invertible for each  $x$ . Furthermore, we let  $T_q \mathcal{M}$ , denote the **tangent space** to  $\mathcal{M}$  at  $q \in \mathcal{M}$ . In this regard, the space  $T\mathcal{M}$  given by  $\cup_{q \in \mathcal{M}} T_q \mathcal{M}$  is called the **tangent bundle**. As a result of the fact that  $T_q \mathcal{M}$  is a linear space, we can find its dual  $T_q^* \mathcal{M}$  known as the **cotangent space** at  $q \in \mathcal{M}$ . The space  $T^* \mathcal{M}$  corresponding to  $\cup_{q \in \mathcal{M}} T_q^* \mathcal{M}$ , is called the **cotangent bundle**. We suppose that  $x \rightarrow a(x) + b(x)u$  is **complete** for each  $u \in \mathbf{R}^m$ . This means that there exists a unique solution of  $\dot{x} = a(x) + b(x)u$  for all  $t > 0$  for any initial condition  $x(0) = x_1 \in \mathcal{X}$ . Hence, given any initial condition  $x_1 \in \mathcal{X}$ , the system of (1) induces a well-defined causal input–output map  $T_\Sigma^{x_1} : L_{2,e}^m[0, \infty) \mapsto L_{2,e}^p[0, \infty)$ . Operators  $T_\Sigma^{x_1}$  arising in this way from a system of (1) are automatically **causal**, i.e.,  $T_\Sigma^{x_1}[u](t) = T_\Sigma^{x_1}[u'](t)$  for  $t \leq T$  whenever  $u(t) = u'(t)$  for  $t \leq T$ . We assume that  $T_\Sigma^{x_1}$  extends by continuity to define a well-defined map on  $L_{2,e}^m[0, \infty)$  (the space of measurable  $\mathbf{R}^m$ -valued functions  $u(\cdot)$  on  $[0, \infty)$  such that  $\int_0^T \|u(t)\|^2 dt < \infty$  for all  $T < \infty$ ). We only consider systems  $\Sigma$  that are **stable**. In other words, at the systems level, the vector field  $x \rightarrow a(x)$  is globally asymptotically stable while  $T_\Sigma^x$  is a diffeomorphism of  $L_2^m[0, \infty)$  for each  $x \in \mathcal{M}$  at the input–output level. If we reduce these notions to the linear case, it means that the transfer function of the system is defined and bounded in the right half-plane, i.e., all poles of the transfer function lie in the left half-plane. The aforementioned stability assumption is made in order to avert the problem of having to find a nonlinear analogue of the filtering algebraic Riccati equation. The square case, corresponds to the situation where  $p = m$ . [10] discusses the case where also  $d(x)$  is invertible for each  $x$ , i.e.,  $\Sigma$  is *invertible*. In particular, this paper investigates the situation where  $d(x)$  is square, invertible and satisfies  $E(x) = d(x)^T d(x) = d(x)d(x)^T$ . We recall that  $d(x)^T d(x) = d(x)d(x)^T$  is a sufficient condition for  $\Sigma$  to be square. On the other hand, at certain instances, this paper involves analyzing nonsquare ( $p \neq m$ ) systems (not invertible). In either situation, the system of (1) is a (state-space) **realization**

of the input–output operator  $T_\Sigma : \mathcal{L}_{2,e}^m[0, \infty) \rightarrow \mathcal{L}_{2,e}^p[0, \infty)$ . We assume that all maps of the type given by  $T_\Sigma$  are Frechet differentiable (see [17]). The Frechet derivative of  $T_\Sigma$  in the above for  $u$  an element of  $\mathcal{L}_{2,e}^m[0, \infty)$ , denoted by  $DT_\Sigma(u)$ , is a linear mapping from  $\mathcal{L}_{2,e}^m[0, \infty)$  to  $\mathcal{L}_{2,e}^p[0, \infty)$ . In this case, the transpose  $[DT_\Sigma(u)]^T$  with respect to the  $\mathcal{L}_2$  inner product exists. The realization  $\Sigma$  is said to be **minimal** if the dimension  $n$  of the state manifold  $\mathcal{M}$  is as small as possible among all possible state space realizations of the given input–output operator  $T_\Sigma$ . In the context of realizations, minimality plays an important role at various levels. For instance, it has a vital part to play when establishing the uniqueness of a realization of a given input–output map (see, for example, Theorem 10). In addition, the major objective of our paper is to characterize *minimal*, stable, spectral factors and provide explicit formulas for them. Here, the assumption that the realizations for most of the nonlinear systems are *minimal* makes computations much easier. In a nonlinear setting, the interpretation of a system of the form  $u \rightarrow [D\Sigma(u)]^T \Sigma(u)$  as a covariance matrix of a stochastic system is less obvious. The realization problem for such systems was solved in [19], where it was postulated that a nonlinear system of this type has a Hamiltonian structure. In this regard, the **Hamiltonian extension** of  $\Sigma$  [where  $\Sigma$  is given as in (1)] has the form

$$\begin{cases} \dot{x} = a(x) + b(x)u \\ \dot{p} = - \left[ \frac{\partial a}{\partial x}(x) + \frac{\partial b}{\partial x}(x)u \right]^T p \\ \quad - \left[ \frac{\partial c}{\partial x}(x) + \frac{\partial d}{\partial x}(x)u \right]^T u_a \\ y = c(x) + d(x)u \\ y_a = b^T(x)p + d^T(x)u_a \end{cases} \quad (2)$$

(see [19]), where  $u, y_a \in \mathbf{R}^m$  and  $u_a \in \mathbf{R}^p$ . Moreover, (2) has a state space equal to the cotangent bundle,  $T^* \mathcal{M}$ , of the state–space manifold  $\mathcal{M}$  with **natural** local coordinates  $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$  with inputs equal to  $(u, u_a)$  and outputs equal to  $(y, y_a)$ . If we impose the interconnection law  $u_a = y$  in (2), we get the Hamiltonian system of the form

$$\Phi = [D\Sigma]^T \circ \Sigma : \begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p, u) \\ \dot{p} = - \frac{\partial H}{\partial x}(x, p, u) \\ y_a = \frac{\partial H}{\partial u}(x, p, u) \end{cases} \quad (3)$$

with Hamiltonian function  $H(x, p, u)$  given by

$$H(x, p, u) = p^T [a(x) + b(x)u] + \frac{1}{2} [c(x) + d(x)u]^T [c(x) + d(x)u]. \quad (4)$$

Here, the state–space is  $T^* \mathcal{M}$ , inputs  $u \in \mathbf{R}^m$  and the outputs  $y_a \in \mathbf{R}^m$ . In (3), the effect of  $[D\Sigma]^T$  in  $[D\Sigma]^T \circ \Sigma$  which produces the output  $y_a$  needs more than just the output  $u_a = y$  if  $\Sigma$  but in addition requires the state  $x$  of the right factor at each time  $t$ . Note that the input–output map  $T_{[D\Sigma]^T \circ \Sigma}$  can be viewed as the composition  $u \rightarrow [DT_\Sigma^{x_1}(u)]^T \circ T_\Sigma^{x_1}(u)$

of the input–output map  $T_{\Sigma}^{x_1}$  with the transpose (in the  $\mathcal{L}_2$ -inner product) of its Frechet derivative  $DT_{\Sigma}^{x_1}(u)$  (evaluated at the same input  $u$ ). We view  $\Phi$  as the nonlinear analogue of a spectral density function exhibited in spectral factorization form  $\Phi(s) = W(-s)^T \cdot W(s)$ . Systems of the type previously described satisfy an energy conservation law that resembles the idea of the conservative system in Willems' theory (see [67]). In turn this is related to the control of mechanical systems (see [43] and [62]). We introduce the notion of the Hamiltonian extension of a nonlinear system in (2) to transform the (all-pass) inner-outer factorization problem into a nonlinear spectral factorization problem (see [64] for more details). In this regard,  $\Sigma$  appearing in (3) is known as a **spectral factor** of  $\Phi$ . In this paper, we discuss a large variety of spectral factors, that, for instance, may be minimal, stable, antistable, minimum phase or maximum phase. Moreover, we introduce an important regularity property of the spectral system  $\Phi$  in (3) in the following way.  $\Phi$  is said to be **weakly coercive** if all its spectral factors are at least one-sided invertible. In particular, in the sequel, we compute right-sided inverses of various extremal spectral factors.

Next, we describe some of the other geometric structure of a realization of the form  $\Phi = [D\Sigma]^T \circ \Sigma$  (see [10]) given as in (2). Before, we discuss this important issue, we provide a characterization of a Lagrangian invariant submanifold. A submanifold  $\mathcal{N} \subset T^*\mathcal{M}$  is called a **Lagrangian submanifold** if  $\dim \mathcal{N} = \dim \mathcal{M}$  and the restriction of the symplectic form  $\omega$  to  $\mathcal{N}$  is zero. In other words, for the vector fields  $F_1(q)$  and  $F_2(q)$  in  $T_q\mathcal{N} \subset T_qT^*\mathcal{M}$  we have that  $\omega(q) = (F_1(q), F_2(q)) = 0$ . Furthermore,  $\mathcal{N}$  is an **invariant manifold** under the vector field  $F$  if  $F(q) \in T_q\mathcal{N}$  for every  $q \in \mathcal{N}$ . Suppose that  $H : T^*\mathcal{M} \rightarrow \mathbf{R}$  is a smooth function that is called a **Hamiltonian function**. Then, the Hamiltonian vector field  $F_H$  on  $T^*\mathcal{M}$  associated with  $H$  is defined by putting  $\omega(F_H, E) = -dH(E)$  for every vector field  $E$  on  $T^*\mathcal{M}$ . Suppose that  $(x_0, p_0) \in T^*\mathcal{M}$  is an equilibrium point for  $F_H$ , or equivalently  $dH(x_0, p_0) = 0$ , then the linearization of  $F_H$  at  $(x_0, p_0)$  is given by a Hamiltonian matrix denoted by  $DF_H(x_0, p_0)$ . Here,  $DF_H(x_0, p_0)$  defines a linear Hamiltonian vector field corresponding to the quadratic Hamiltonian given by the quadratic terms in the Taylor expansion of  $H(x, p)$  around  $(x_0, p_0)$ . Moreover, the manifold  $\mathcal{N}$  is known as a **stable invariant manifold** of  $F_H$  if  $\mathcal{N}$  is tangent at  $(x_0, p_0)$  to the stable eigenspace of  $DF_H(x_0, p_0)$ . Similarly,  $\mathcal{N}$  is known as an **antistable invariant manifold** of  $F_H$  if  $\mathcal{N}$  is tangent at  $(x_0, p_0)$  to the antistable eigenspace of  $DF_H(x_0, p_0)$ .

It is a well-known fact that if  $E(x) := d(x)^T d(x)$  is invertible then  $\Phi = [D\Sigma]^T \circ \Sigma$  is causally invertible. In this case, the inverse system  $\Phi^{-1}$  is also of the Hamiltonian form (3) with Hamiltonian  $H^\times(x, p, y_a)$  equal to the Legendre transform of  $H(x, p, u)$

$$H^\times(x, p, y_a) = H(x, p, u) - u^T y_a$$

$$u \text{ satisfying } y_a = \frac{\partial H}{\partial u}(x, p, u) \quad (5)$$

or, more explicitly

$$H^\times(x, p, y_a) = p^T [a(x) - b(x)E^{-1}(x)d(x)^T c(x)]$$

$$+ \frac{1}{2}c(x)^T [I - d(x)E^{-1}(x)d(x)^T] c(x)$$

$$- \frac{1}{2}p^T b(x)E^{-1}(x)b(x)^T p$$

$$+ [p^T b(x) + c(x)^T d(x)] E^{-1}(x)y_a$$

$$- \frac{1}{2}y_a^T E^{-1}(x)y_a \quad (6)$$

where  $E(x) = d(x)^T d(x)$ . In addition, the Hamiltonian system that corresponds to  $\Phi^{-1}$  is given by

$$\Phi^{-1} = ([D\Sigma]^T \circ \Sigma)^{-1} : \begin{cases} \dot{x} = \frac{\partial H^\times}{\partial p}(x, p, y_a) \\ \dot{p} = -\frac{\partial H^\times}{\partial x}(x, p, y_a) \\ u = -\frac{\partial H^\times}{\partial y_a}(x, p, y_a). \end{cases} \quad (7)$$

In this case, the submanifold  $\mathcal{N}^\times = \{(x, p) \in T^*\mathcal{M} : p = 0\}$  is a stable invariant Lagrangian manifold for  $\Phi^{-1}$  with  $y_a = 0$ . On the other hand, the submanifold  $\mathcal{N} = \{(x, p) \in T^*\mathcal{M} : x = 0\}$  is the antistable invariant manifold for  $\Phi$  with  $u = 0$  and is Lagrangian.

### B. Linear Case

The results that we establish in our analysis in subsequent sections of this paper may be regarded as natural nonlinear analogues of some of those obtained in [28], [39], and [50]–[52] for rational matrix functions. In particular, the explicit formulas for the spectral factors derived in the latter sections bear a resemblance to those in [28] and [51], where the formulas are written in terms of solutions of algebraic Riccati equations. Later, we provide a brief discussion of the pertinent linear factorization results for our purposes.

We denote the subspace of  $\mathcal{L}^2$  of functions analytic in the right half-plane with sup norm by  $H_+^\infty$ .  $H_-^\infty$  is defined similarly on the left half-plane. In the linear case, for a system  $\Sigma$  given by the state equations

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, & u \in \mathbf{R}^m \\ y = Cx + Du, & y \in \mathbf{R}^p \end{cases} \quad (8)$$

$[D\Sigma]^T \Sigma$  is the series interconnection of  $\Sigma$  and the adjoint system  $\Sigma^T$  having the state–space equations

$$\Sigma^T : \begin{cases} \dot{p} = -Ap - C^T u_a, & u_a \in \mathbf{R}^m \\ y_a = B^T p + D^T u_a, & y_a \in \mathbf{R}^p. \end{cases} \quad (9)$$

This scheme has an associated transfer matrix  $W(-\lambda)^T W(\lambda)$ , where

$$W(\lambda) = D + C(\lambda I - A)^{-1}B. \quad (10)$$

We say that a  $p \times m$  proper rational matrix function  $W$  is a **spectral factor** if  $\Phi = W^*W$ , where  $W^*(\lambda) = W(-\bar{\lambda})^*$ . Furthermore, spectral factor  $W$  is called **stable (antistable)** if  $W \in H_+^\infty$  ( $W \in H_-^\infty$ ). Linear spectral factors of matrix functions were parametrized in several ways (see [22], [27], [28], [40], [50]–[53], and [57]). In particular, in [51] it was shown that one may determine a parametrization of all minimal square spectral factors of a positive semidefinite rational matrix function in terms of invariant subspaces (see also [50], [57], and

[58]), minimal unitary left divisors of a certain unitary function (see also [28] and [52]) and algebraic Riccati equations (see [40] and [57]; also [47], [50], [58], and [66]). Reference [28] approaches the problem from a geometric viewpoint that involves shift invariant subspaces. In particular, state space formulas for matrix functions that arise from the analysis of singular, i.e., rectangular and not necessarily full rank, spectral factors, are derived. The approaches in [47] and [45] are related to the methods used in [28] because they emphasize stable spectral factors with  $\Phi$  having some regularity properties on  $i\mathbf{R}$ . A full-column rank  $p \times m$  rational matrix function  $W$  in  $H_+^\infty$  is said to be **minimum phase** or **outer** (on the right) if  $\text{rank } W(\lambda) = m$  for  $\text{Re } \lambda > 0$ . A system is **maximum phase** if  $\text{rank } W(\lambda) = m$  for  $\text{Re } \lambda < 0$ .

Let the nonsquare spectral factor  $W_1$  of the spectral function  $\Phi$ , given by

$$W_1(\lambda) = D_1 + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (\lambda I - A)^{-1} B \quad (11)$$

be a  $p \times m$  minimal, stable spectral factor. In order to obtain an appropriate form for  $D_1$  of  $W_1$  in (11) we assume that  $W(\lambda)$  in (10) is square and has full-rank almost everywhere. So, also,  $D$  has full-rank without loss of generality. Next, we can apply some standard systems theory in order to write  $D_1$  in (11) as  $D_1 = U \begin{pmatrix} D \\ 0 \end{pmatrix} V$  with invertible  $U$  and  $V$ . We may put the singular values either in the columns of  $U$  or in the rows of  $V$ . Thus, we may assume that  $U$  (or  $V$ ) is unitary if so needed. In this case, if we have that  $\Phi = W^*W$  then it follows that  $U\Phi U^* = W_1^*W_1$ . We are especially interested in the case where  $V = U = I$ , i.e.,  $\Phi = W^*W = W_1^*W_1$ , with  $D_1$  in (11) having the form

$$D_1 = \begin{pmatrix} D \\ 0 \end{pmatrix}. \quad (12)$$

The argument for obtaining the  $D_1$ -term for a nonsquare spectral factor, say  $W_1$ , from the  $D$ -term for a full rank square spectral factor, say  $W$ , can easily be extended to the nonlinear case using the development in [44]. In particular, in the sequel, this argument is needed for Theorem 3.

Furthermore, we can describe the minimal realizations of the stable minimum- and maximum phase, spectral factors  $W_-$  and  $W_+$ , respectively, of  $\Phi$  as follows.

A minimal realization of the stable, minimum phase, spectral factor  $W_-$  of  $\Phi$  has the form

$$W_-^*(\lambda) = D + C(\lambda I - A)^{-1} (B_1 + X_- C^* (D^* D)^{-1}) \quad (13)$$

where  $X_- \geq 0$  is the stabilizing solution of the algebraic Riccati equation

$$\begin{aligned} & (A - B_1(D^* D)^{-1} D^* C) X_- + X_- (A^* - C^* D(D^* D)^{-1} B_1^*) + \\ & B_2 B_2^* - X_- C^* D(D^* D)^{-2} D^* C X_- = 0. \end{aligned} \quad (14)$$

A minimal realization of the stable, maximum phase, spectral factor  $W_+$  of  $\Phi$  has the form

$$W_+^*(\lambda) = D + C(\lambda I - A)^{-1} (B_1 + X_+ C^* (D^* D)^{-1}) \quad (15)$$

where  $X_- \geq 0$  is the antistabilizing solution of the algebraic Riccati equation

$$\begin{aligned} & (A - B_1(D^* D)^{-1} D^* C) X_+ + X_+ (A^* - C^* D(D^* D)^{-1} B_1^*) + \\ & B_2 B_2^* - X_+ C^* D(D^* D)^{-2} D^* C X_+ = 0. \end{aligned} \quad (16)$$

In the linear case, the function  $\Theta_{\text{linear}}$  is said to be *column rigid* if  $\Theta_{\text{linear}}^* \Theta_{\text{linear}} = I$  &  $p \geq m$  and is *row rigid* if  $\Theta_{\text{linear}} \Theta_{\text{linear}}^* = I$  &  $p \leq m$ . A minimal column rigid function  $Q_1$  satisfying  $W_-^* Q_1 = W^*$  has the form

$$\begin{aligned} Q_1(\lambda) &= (I \quad 0) + (D^* D)^{-1} D^* C \\ &\times \left[ \lambda I - \left( A - \left( B_1 + X_- C^* D(D^* D)^{-2} D^* C \right) \right) \right]^{-1} \\ &\times \begin{pmatrix} -X_- C^* D(D^* D)^{-1} & B_2 \end{pmatrix}. \end{aligned}$$

A minimal row rigid function  $Q_2$  satisfying  $W^* Q_2 = W_+^*$  has the form

$$\begin{aligned} Q_2(\lambda) &= \begin{pmatrix} I \\ 0 \end{pmatrix} + \begin{pmatrix} (D^* D)^{-1} D^* C X_+ \\ -B_2^* \end{pmatrix} \\ &\times \left[ \lambda I - \left( -A^* - C^* D(D^* D)^{-1} \right. \right. \\ &\quad \left. \left. \times (B_1^* + C^* D(D^* D)^{-2} D^* C X_+^*) \right) \right]^{-1} \\ &\times C^* D(D^* D)^{-1}. \end{aligned}$$

By using the fact that the analysis involving the solution  $P$  of the Hamilton–Jacobi inequality and equality, (30) and (31), respectively, may be reduced to the analysis involving the solutions  $X$  of an algebraic Riccati equation in the linear case, Theorem 3 of this paper reduces to [51, Th. 2.2]. For ease of comparison we formulate the statement of the latter theorem shown later. Suppose that a positive–semidefinite rational matrix function  $\Phi$  has a realization  $\Phi(\lambda) = I_m + C(\lambda I - A)^{-1} B$ . There is a one-to-one correspondence between the set of minimal nonsquare spectral factors  $W(\lambda)$  of  $\Phi(\lambda)$  such that  $W(\infty) = (I_m \ 0)$  and the set of triples  $\{\mathcal{M}, X, \hat{B}_1\}$ . Here,  $\mathcal{M}$  is an  $A$ -invariant  $H$ -Lagrangian subspace. To describe  $X$  and  $\hat{B}_1$ , let  $A_1$  and  $C_1$  be given by  $A_1 = A|_{\mathcal{M}}$  and  $C_1 = C|_{\mathcal{M}}$ . Furthermore, suppose that  $\mathcal{M}^\times$  is the  $A^\times = (A - BC)$ -invariant,  $H$ -Lagrangian subspace such that  $\sigma(A^\times|_{\mathcal{M}^\times}) \subset \overline{\mathcal{C}}_-$ . Let  $\pi$  be the projection onto  $\mathcal{M}$  along  $\mathcal{M}^\times$  and denote a matrix representation for  $\pi B$  by  $\tilde{B}_1$ . Then  $X$  solves the Riccati inequality  $X C_1^* C_1 X - X(A_1 - \tilde{B}_1 C_1)^* - (A_1 - \tilde{B}_1 C_1) X \leq 0$  and  $\hat{B}_1$  satisfies  $X C_1^* C_1 X - X(A_1 - \tilde{B}_1 C_1)^* - (A_1 - \tilde{B}_1 C_1) X = -\hat{B}_1 \hat{B}_1^*$ . This correspondence is given by

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1} \begin{pmatrix} X C_1^* + \tilde{B}_1 & \hat{B}_1 \end{pmatrix}.$$

### C. Nonlinear Spectral Factorization Problem Formulation

We are now in a position to formally state the problem that we will investigate in the sequel.

Given a stable, nonlinear, square, input-affine system  $\Sigma$  as before, describe certain nonsquare, stable, nonlinear, input-affine systems

$$R: \begin{cases} \dot{\bar{x}} = \bar{a}(\bar{x}) + \bar{b}(\bar{x})u, & u \in \mathbf{R}^m \\ y_a = \bar{c}(\bar{x}) + \bar{d}(\bar{x})u, & y_a \in \mathbf{R}^p \end{cases} \quad (17)$$

(with state-space manifold equal to  $\bar{\mathcal{M}}$  having local coordinate  $\bar{x} \in \mathbf{R}^n$  and with  $T^*\bar{\mathcal{M}}$  having local coordinates  $(\bar{x}, \bar{p}) \in \mathbf{R}^{2n}$ ) so that  $\Phi = [D\Sigma]^T \circ \Sigma = [DR]^T \circ R$ , or, at the input-output level, for each  $x_1 \in \mathcal{M}$  there is a unique  $\bar{x}_1 \in \bar{\mathcal{M}}$  so that

$$[DT_\Sigma^x(u)]^T \circ T_\Sigma^{x_1}(u) = [DT_R^{\bar{x}_1}(u)]^T \circ T_R^{\bar{x}_1}(u)$$

for all  $u \in \mathcal{L}_2^m[0, \infty)$ . We note that by comparison with [10], in our case  $\bar{d}(x)$  need not necessarily be invertible. The geometric structure of  $R$  given by (17) with input-output map  $T_R$  acting on  $\mathcal{L}_{2,e}^m[0, \infty)$  may be laid down as follows.  $\Phi = [DR]^T \circ R$  is a Hamiltonian system (compare (3)) with state space equal to the cotangent bundle  $T^*\bar{\mathcal{M}}$  of the state manifold  $\bar{\mathcal{M}}$  for  $R$  with Hamiltonian function  $H_R(\bar{x}, \bar{p}, u)$  given by

$$H_R(\bar{x}, \bar{p}, u) = \bar{p}^T [\bar{a}(\bar{x}) + \bar{b}(\bar{x})u] + \frac{1}{2} [\bar{c}(\bar{x}) + \bar{d}(\bar{x})u]^T [\bar{c}(\bar{x}) + \bar{d}(\bar{x})u].$$

Under the assumption that  $\bar{E}(x) = \bar{d}(x)^T \bar{d}(x)$  is invertible we know that the inverse system  $\Phi^{-1} = ([DR]^T \circ R)^{-1}$  is also of the Hamiltonian form (7) with Hamiltonian  $H_R^\times(\bar{x}, \bar{p}, y_a)$  equal to the Legendre transform of  $H_R(\bar{x}, \bar{p}, u)$

$$H_R^\times(\bar{x}, \bar{p}, y_a) = H(\bar{x}, \bar{p}, u) - u^T y_a$$

$u \text{ satisfying } y_a = \frac{\partial H_R}{\partial u}(\bar{x}, \bar{p}, u)$

or, more explicitly

$$\begin{aligned} H^\times(\bar{x}, \bar{p}, y_a) = & \bar{p}^T [\bar{a}(\bar{x}) - \bar{b}(\bar{x})\bar{E}^{-1}(\bar{x})\bar{d}(\bar{x})^T \bar{c}(\bar{x})] \\ & + \frac{1}{2} \bar{c}(\bar{x})^T [I - \bar{d}(\bar{x})\bar{E}^{-1}(\bar{x})\bar{d}(\bar{x})^T] \bar{c}(\bar{x}) \\ & - \frac{1}{2} \bar{p}^T \bar{b}(\bar{x}) \bar{E}^{-1}(\bar{x}) \bar{b}(\bar{x})^T \bar{p} \\ & + [\bar{p}^T \bar{b}(\bar{x}) + \bar{c}(\bar{x})^T \bar{d}(\bar{x})] \bar{E}^{-1}(\bar{x}) y_a \\ & - \frac{1}{2} y_a^T \bar{E}^{-1}(\bar{x}) y_a. \end{aligned} \quad (18)$$

The submanifold  $\bar{\mathcal{N}} = \{(\bar{x}, \bar{p}) \in T^*\bar{\mathcal{M}} : \bar{x} = 0\}$  is the anti-stable invariant manifold for  $\Phi$  with  $u = 0$  and is Lagrangian. The submanifold  $\bar{\mathcal{N}}^\times = \{(x, p) \in T^*\bar{\mathcal{M}} : \bar{p} = 0\}$  is a Lagrangian invariant manifold for the system  $\Phi^{-1}$  with  $y_a = 0$ . We note that when referring to (18) the term ‘‘Legendre’’ may be replaced by the term ‘‘Young’’ (see [38] and the references therein for further information). In other words, in our case, the argument will hold equally well if we consider the Young transforms instead of the Legendre transforms.

As was mentioned before, spectral factorization is intimately related to inner-outer factorization. Since *nonlinear inner systems* play a role in the ensuing analysis (see, for instance, Lemma 8 and Theorem 9), we provide a brief description

of such a nonlinear system. Nonlinear  $\Theta$  is **inner** (or **stable conservative**) if  $x \rightarrow a(x)$  is stable (w.r.t. assumed equilibrium point  $x = 0$ ) and there is a nonnegative-valued storage function  $P(x)$  with  $P(0) = 0$  such that

$$P(x(t_2)) - P(x(t_1)) = \frac{1}{2} \int_{t_1}^{t_2} [u(t)^T u(t) - y(t)^T y(t)] dt \quad (19)$$

over all trajectories  $(u(t), x(t), y(t))$  of the system. This is true for all  $t_1 \leq t_2$  and  $u(\cdot)$ , with  $x(t_2)$  denoting the state at time  $t_2$  originating from the initial state  $x(t_1)$  at time  $t_1$  and input  $u(\cdot)$  on the time interval  $[t_1, t_2]$ . Alternatively,  $\Theta$  is said to be **inner** if it is lossless with respect to the  $L_2$ -gain supply rate  $s(u, y) = (1/2)u^T u - (1/2)y^T y$ . The above characterization of nonlinear inner systems, was achieved within the dissipative systems framework of Hill–Moylan–Willems (see [33] and [66]). Here the dissipation equality in (19) may be derived from a state space-implementation of the  $L_2$ -gain condition in the formulation of the nonlinear  $H_\infty$ -problem (see [63] and [64]). Note that the function defined in (19) may also be thought of as a Lyapunov function (see [33]). If  $P$  is assumed to be smooth, the energy balance relation (19) can be expressed as

$$P_x(x) [a(x) + b(x)u] = \frac{1}{2} u^T u - \frac{1}{2} [c(x) + d(x)u]^T \times [c(x) + d(x)u], \quad P(0) = 0$$

for all  $x$  and  $u$  or, equivalently, in infinitesimal form as

$$\begin{aligned} P_x(x)b(x) + c(x)^T d(x) &= 0 \\ P_x(x)a(x) + \frac{1}{2} c(x)^T c(x) &= 0 \\ d(x)^T d(x) &= I. \end{aligned}$$

Although some of the concepts mentioned in this paragraph have not yet been formally defined, a comment on notation is in order at this point. Throughout this paper,  $\Sigma$  denotes a square nonlinear system that is a spectral factor with the property that  $E(x) = d(x)^T d(x)$  is invertible. On the other hand,  $R$  is a nonsquare spectral factor that is the starting point for our analysis in the sense that subsequent explicit formulas for various types of spectral factors and inner systems are written in terms of its components and associated Hamilton–Jacobi equation. Given a general spectral system  $\Phi$  we denote by  $\Sigma_-$  and  $\Sigma_+$ , the minimal, stable, minimum and maximum phase spectral factors, respectively. In fact, as for the case of regular, square spectral factors in [10] a range of *extremal spectral factors* like  $\Sigma_-$  (minimal, stable, minimum phase),  $\Sigma_+$  (minimal, stable, maximum phase),  $\tilde{\Sigma}_-$  (minimal, antistable, minimum phase) and  $\tilde{\Sigma}_+$  (minimal, antistable, maximum phase) form the basis for our analysis of all square and nonsquare spectral factors ([10] for the square case).  $\Theta_c$  and  $\Theta_r$  is the notation used for column rigid and row rigid systems, respectively (see Section IV-B for a full characterization of rigid systems).  $\Phi$  is the symbol used to denote the nonlinear (spectral) system that undergoes spectral factorization of the type given in (3).

#### D. Novel Features and Structure of This Paper

Some of the novel features of this paper are given as follows.

- In Proposition 1 of Section III, we derive a nonsquare spectral factor from a square one by comparing the components of their respective Hamiltonian systems (and corresponding Hamiltonian functions). In particular, this proposition provides a necessary and sufficient condition for a (square or nonsquare) nonlinear system to be a spectral factor in the case where our starting point is a given square spectral factor.
- In the main result of this paper, we establish a bijective correspondence between nonsquare, stable, nonlinear, spectral factors  $R$  of  $\Phi$  of the form (17) and the set of triples  $(\mathcal{X}, P, c_2)$ . Here,  $\mathcal{X}$  is an invariant Lagrangian submanifold of  $\Phi^{-1}$  where  $y_a = 0$ ,  $P$  is a smooth solution of a certain Hamilton–Jacobi inequality and  $c_2$  is a component of  $R$  with special properties (see Theorem 3 in Section III).
- In Proposition 5 of Section IV, we verify the existence of stable, minimum and maximum phase spectral factors of  $\Phi$  that are related to the square spectral factor  $\Sigma$  in (1).
- We determine connections between the nonsquare spectral factor  $R$  of the form given by (17) and the extremal spectral factors mentioned before. For instance, in Theorem 6 we express stable, minimum phase and stable, maximum phase spectral factors explicitly in terms of the components of the nonsquare spectral factor  $R$ . Also, in Theorem 7, the right inverses of the aforementioned extremal spectral factors are computed. Lemma 8 and Theorem 9 establish useful relationships between nonlinear extremal spectral factors and inner systems.
- In Section IV-B, we introduce the notions of column and row rigid systems. In addition, we consider the connection between rigid systems, the extremal spectral factors and the nonsquare spectral factor  $R$  (see Theorem 10). Explicit formulas for these rigid systems are provided in Theorem 11.

Next, we briefly outline other issues that arise in this paper. In Section II-B, we make connections between literature dealing with linear spectral factorization problems and the theory developed in our paper. Section III provides remarks relating Theorem 3 to pre-existing papers that discuss the square nonlinear case (see Remark 4). The main technique that we will employ to prove our results in the sequel involves a comparison between the Hamiltonian functions of the Hamiltonian system corresponding to the spectral factors for which we would like to establish some properties. This usually is accompanied by an appropriately chosen change of coordinates. This method of proof underlies the verification of Proposition 1, Theorem 3, Theorem 6, Lemma 8, and Theorem 10.

### III. PARAMETRIZATION IN TERMS OF LAGRANGIAN MANIFOLDS AND HAMILTON–JACOBI INEQUALITIES

The next result attempts to relate the components of stable, nonlinear, square, input-affine systems given by  $\Sigma$  in (1) to nonsquare, stable, nonlinear, input-affine systems given by  $R$  in

(17). Recall from the discussion on the geometric structures of  $\Sigma$  and  $R$  that  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  are the  $n$ -dimensional state manifolds of  $\Sigma$  and  $R$ , respectively. In order to simplify notation note that by the requirement for  $R$  that for all  $x_1 \in \mathcal{M}$  there exists a unique  $\bar{x}_1 \in \bar{\mathcal{M}}$  we can identify  $\bar{\mathcal{M}}$  with  $\mathcal{M}$ , and  $\bar{x}$  with  $x$ .

*Proposition 1:* Assume that  $\Phi = [D\Sigma]^T \circ \Sigma = [DR]^T \circ R$ , where  $\Sigma$  and  $R$  are given as in (1) and (17), respectively, with Hamiltonian functions given by (4) and (18), respectively. Furthermore, suppose that  $E(x) = d(x)^T d(x) = d(x)d(x)^T$  is invertible. In this case, we have that

$$\bar{a}(x) = a(x) \quad (20)$$

$$\bar{b}(x) = b(x) \quad (21)$$

$$\frac{1}{2}\bar{c}(x)^T \bar{c}(x) = P_x(x)a(x) + \frac{1}{2}c(x)^T c(x) \quad (22)$$

$$\bar{c}(x)^T \bar{d}(x) = P_x(x)b(x) + c(x)^T d(x) \quad (23)$$

$$\bar{d}(x)^T \bar{d}(x) = d(x)^T d(x) \quad (24)$$

for some smooth function  $P$  on  $\mathcal{M}$ . Conversely, for any smooth function  $P$  on  $\mathcal{M}$  the previous equations define a system  $R$ .

*Proof:* In the proof, we follow the scheme suggested in [13] in order to show that a solution of the Hamilton–Jacobi equation corresponds to a spectral factor. The argument is as follows. By assumption,  $\Phi = [D\Sigma]^T \circ \Sigma$  has a state space realization of the form (3) with Hamiltonian  $H$  given by (4). On the other hand, if  $R$  is given by (17) and  $\Phi = [DR]^T \circ R$ , then  $\Phi$  has the Hamiltonian form (3) but with  $H_R$  given by (18) in place of  $H$ . Under the assumption that  $P$  is a smooth function, it follows that  $(x, \bar{p}) = (x, P_x^T(x) + p)$  is a canonical change of coordinates (see [2], [13], or [10]), transforming the Lagrangian submanifold  $\{(x, p) \in T^*\mathcal{M} : p = -P_x^T(x)\}$  into  $\bar{\mathcal{N}}^\times = \{(x, \bar{p}) \in T^*\mathcal{M} : \bar{p} = 0\}$ . Conversely, for any choice of local canonical coordinates  $(x, \bar{p})$  for  $T^*\mathcal{M}$  there exists (locally) a smooth function  $P$  such that  $\{(x, \bar{p}) \in T^*\mathcal{M} : \bar{p} = 0\} = \{(x, p) \in T^*\mathcal{M} : p = -P_x^T(x)\}$ . The main requirement for the completion of the proof is to express  $H$  in terms of the new coordinates  $H(x, P_x(x)^T + p)$  and then identify  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$ ,  $\bar{d}$  so as to ensure that  $H(x, P_x(x)^T + p) = H_R(x, p)$ . Calculation shows that

$$\begin{aligned} H(x, P_x(x)^T + p) &= (P_x(x)^T + p)^T (a(x) + b(x)u) \\ &\quad + \frac{1}{2}(c(x) + d(x)u)^T (c(x) + d(x)u) \\ &= p^T (a(x) + b(x)u) + P_x(x)a(x) \\ &\quad + \frac{1}{2}c(x)^T c(x) \\ &\quad + (P_x(x)b(x) + c(x)^T d(x))u \\ &\quad + \frac{1}{2}u^T d(x)^T d(x)u. \end{aligned} \quad (25)$$

We are required to define  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  and  $\bar{d}$  so that  $H(x, P_x(x)^T + p) = H_R(x, \bar{p})$ , where

$$\begin{aligned} H_R(x, \bar{p}) &= \bar{p}^T (\bar{a}(x) + \bar{b}(x)u) + \frac{1}{2}\bar{c}(x)^T \bar{c}(x) \\ &\quad + \bar{c}(x)^T \bar{d}(x)u + \frac{1}{2}u^T \bar{d}(x)^T \bar{d}(x)u. \end{aligned} \quad (26)$$

Since  $[D\Sigma]^T \circ \Sigma = [DR]^T \circ R$ , we know that (25) must correspond to (26). Hence it is easy to see that (20)–(24) are satisfied.  $\blacksquare$

The proof of Proposition 1 is dependent on an important part of the proof of [10, Th. 2.3], where square, stable spectral factors are considered. The said proposition will be used extensively in the sequel (see, for instance, Theorems 3 and 6).

*Remark 2:* Proposition 1 and the related discussion on the geometric structure of  $\Sigma$  and  $R$  in the preceding section are suggestive of the fact that Lagrangian invariant manifolds and nonlinear spectral factors are intimately connected. In turn, these notions can also be shown to be related to Hamilton–Jacobi equations in the following way. A method for finding a Lagrangian invariant manifold  $\mathcal{X}$  for  $\Phi^{-1}$  with  $y_a = 0$  is to consider the Hamilton–Jacobi equation  $H^\times(x, P_x^T(x), 0) = 0$ , where  $H^\times(x, p, y_a)$  is given as in (5) and (6) and  $P$  is some smooth function with  $P(0) = 0$  (compare the statement of Proposition 1). If there exists such a  $P$  then the said manifold  $\mathcal{X}$  has the form

$$\{(x, p) \in T^*\mathcal{M} : p = P_x^T(x)\}. \quad (27)$$

The main result that follows shows that certain nonlinear minimal nonsquare stable spectral factor can be expressed completely in terms of a square minimal stable spectral factors and solutions of a certain type of Hamilton–Jacobi equation. The proof of this result has a heavy reliance on the analysis in Proposition 1.

*Theorem 3:* Suppose for  $\Sigma$  in (1) that  $E(x) = d(x)^T d(x) = d(x)d(x)^T$  is invertible and the spectral system  $\Phi = [DS]^T \circ \Sigma$  is given by (3). Then there is a bijective correspondence between nonsquare, stable, nonlinear systems  $R$  given by

$$R : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} + \begin{pmatrix} d(x) \\ 0 \end{pmatrix} u \end{cases} \quad (28)$$

where

$$c_1(x) = c(x) + E^{-1}(x)d(x)b(x)^T P_x(x)^T \quad (29)$$

which provide a spectral factorization  $\Phi = [DR]^T \circ R$  for  $\Phi$  and the set of triples  $(\mathcal{X}, P, c_2)$ . Here,  $\mathcal{X}$  is an invariant Lagrangian submanifold of  $\Phi^{-1}$  where  $y_a = 0$ ,  $P$  is a smooth solution of the Hamilton–Jacobi inequality

$$\begin{aligned} & P_x(x) [a(x) - b(x)d(x)^T E^{-2}(x)d(x) (c_1(x) - c(x))] \\ & - \frac{1}{2}c(x)^T E^{-1}(x)d(x)b(x)^T P_x(x)^T \\ & - \frac{1}{2}P_x(x)b(x)d(x)^T E^{-1}(x)c(x) \\ & + \frac{1}{2}P_x(x)b(x)d(x)^T E^{-2}(x)d(x)b(x)^T P_x(x)^T \\ & \geq 0 \end{aligned} \quad (30)$$

and  $c_2$  satisfies the equation

$$\begin{aligned} & P_x(x) [a(x) - b(x)d(x)^T E^{-2}(x)d(x) (c_1(x) - c(x))] \\ & - \frac{1}{2}c(x)^T E^{-1}(x)d(x)b(x)^T P_x(x)^T \\ & - \frac{1}{2}P_x(x)b(x)d(x)^T E^{-1}(x)c(x) \\ & + \frac{1}{2}P_x(x)b(x)d(x)^T E^{-2}(x)d(x)b(x)^T P_x(x)^T \\ & - \frac{1}{2}c_2(x)^T c_2(x) = 0 \end{aligned} \quad (31)$$

where  $c_2(x)^T c_2(x) \geq 0$ . Moreover, if  $P$  is any solution of the Hamilton–Jacobi equation (31) then there exists a map  $X$  such that

$$c_2(x) = -X_x(x)^T P_x(x)^T. \quad (32)$$

*Proof:* Proposition 1 and the discussion in Section II suggest that the Lagrangian invariant manifold  $\mathcal{X}$  of  $\Phi^{-1}$  where  $y_a = 0$  given by (27) and the nonlinear spectral factors  $\Sigma$  and  $R$  are related. Also, from Remark 2 we have that a relationship exists between these notions and Hamilton–Jacobi inequalities (and ultimately Hamilton–Jacobi equations) (see [13] and [64] for more information). From (24) in Proposition 1 where  $\bar{d}(x)^T \bar{d}(x) = d(x)^T d(x)$ , without loss of generality, we may choose

$$\bar{d}(x) = \begin{pmatrix} d(x) \\ 0 \end{pmatrix}. \quad (33)$$

Putting  $\bar{c}(x) = \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}$ , it is clear from (23) that  $c_1(x) = c(x) + E^{-1}(x)d(x)b(x)^T P_x(x)^T$ . Finally, from this and (22), we have that

$$\begin{aligned} & P_x(x) [a(x) - b(x)d(x)^T E^{-2}(x)d(x)b(x)^T P_x(x)^T] \\ & - \frac{1}{2}c(x)^T d(x)E^{-1}(x)b(x)^T P_x(x)^T \\ & - \frac{1}{2}P_x(x)b(x)E^{-1}(x)d(x)^T c(x) \\ & + \frac{1}{2}P_x(x)b(x)E^{-1}(x)b(x)^T P_x(x)^T \\ & = \frac{1}{2}c_2(x)^T c_2(x). \end{aligned}$$

For  $c_2(x)^T c_2(x) \geq 0$ , (30) and (31) will hold. Furthermore, we note that the  $R$  that we obtain before is stable since the drift-dynamics vector field  $\bar{a}(x)$  for  $R$  is equal to the drift-dynamics vector field  $a(x)$  for  $\Sigma$  and  $\Sigma$  is assumed to be stable. The fact that (32) holds, follows directly from the form of the Hamilton–Jacobi equation (31). ■

The generality of the choice of  $\bar{d}$  in (33) can easily be justified by applying an analogous argument as that for (full-rank) linear spectral factors provided in Section II-B. The reasoning in Theorem 3 follows the discussion in the main result of [64, Sec. 6.3] rather closely. Indeed, if we consider the Hamilton–Jacobi inequality [64, (6.60)] and compute and substitute the minimizing  $u$ , we obtain inequality (30) in Theorem 3.

*Remark 4:* Our analysis suggests that the minimal nonlinear nonsquare stable spectral factors described in Theorem 3 can be reduced to the *square* case. For instance, in [10] it is postulated that there is a one-to-one correspondence between certain square, stable nonlinear systems  $S$  which provide a spectral factorization  $\Phi = [DS]^T \circ S$  for  $\Phi$  and Lagrangian, invariant submanifolds of  $\Phi^{-1}|_{y_a=0}$  which are parametrizable by the base manifold  $\mathcal{M}$  of  $T^*\mathcal{M}$  or, equivalently, smooth solutions  $P$  of the Hamilton–Jacobi equation

$$\begin{aligned} & P_x(x) [a(x) - b(x)E^{-1}(x)d(x)^T c(x)] \\ & + \frac{1}{2}c(x)^T [I - d(x)E^{-1}(x)d(x)^T] c(x) \\ & - \frac{1}{2}P_x(x)b(x)E^{-1}(x)b(x)^T P_x(x)^T = 0. \end{aligned}$$



Moreover, all minimal stable square spectral factors are of the form

$$R : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) + d(x)E^{-1}(x)b(x)^T P_x(x)^T + u. \end{cases}$$

This corresponds exactly with the square stable spectral factor determined in [10, Th. 2.3].

#### IV. RELATIONSHIPS WITH OTHER CONTROL SYSTEMS

In this section, we provide an alternative characterization of *minimum* and *maximum phase* spectral factors and introduce the notions of *column* and *row rigid* systems. In particular, we explain how explicit state-space formulas for these systems may be determined in terms of the formula (28) for  $R$  and special classes of solutions of Hamilton–Jacobi equations.

##### A. Minimum and Maximum Phase Systems

In the literature, it is usual to define a system as being *minimum phase*, for instance, when the dynamics of its inverse system is Lyapunov stable. However, since the classes of spectral factors that we consider in this paper are not necessarily invertible we need to find an alternative description for minimum and maximum phase systems. The following discussion deals with these notions. We recall that the output nulling (or zero) dynamics of a system  $\Sigma$  is the set of all system trajectories  $x(\cdot)$  generated by the some input trajectory  $u(\cdot)$  such that  $y(\cdot)$  is identically zero. Under some regularity conditions it can be computed as

$$\dot{x} = f(x, u), \quad x \in \mathcal{N}^* \subset \mathcal{X}. \quad (34)$$

Under additional conditions, it will actually be a system without inputs  $\dot{x} = f(x)$ ,  $x \in \mathcal{N}^* \subset \mathcal{X}$ . In this case, the system is **minimum phase** if  $f(x)$  is asymptotically stable and **maximum phase** if  $f(x)$  is antistable. For (34), we define the system to be *minimum phase* if there exists  $u = \alpha(x)$  such that the system is stable and maximum phase if there exists  $u = \alpha(x)$  such that the system is antistable. Our first result informs us that it is always possible to derive a stable, minimum (maximum) phase spectral factor  $\Sigma_-$  ( $\Sigma_+$ ) from a spectral factor  $\Sigma$  of the type given by (1).

**Proposition 5:** Suppose that  $\Phi = [D\Sigma]^T \circ \Sigma$ , where  $\Sigma$  is given as in (1) with Hamiltonian function given by (4) and  $E(x) = d(x)^T d(x)$  being invertible. Then, the following statements hold.

- 1) There exists a stable, minimum phase spectral factor  $\Sigma_-$  such that

$$\Phi = [D\Sigma]^T \circ \Sigma = [D\Sigma_-]^T \circ \Sigma_-. \quad (35)$$

- 2) There exists a stable, maximum phase spectral factor  $\Sigma_+$  such that

$$\Phi = [D\Sigma]^T \circ \Sigma = [D\Sigma_+]^T \circ \Sigma_+. \quad (36)$$

*Proof:*

- 1) For the outer spectral factor  $\Sigma_-$ , we have that

$$[D\Sigma_-]^T \circ \Sigma_- \quad (37)$$

has a form of the type given as in (3) with some associated Hamiltonian function  $H_-(x_-, p_-, u)$ . We note from [13] that we can compute the antistable invariant manifold of (37) for  $u = 0$ . Since  $E(x) = d(x)^T d(x)$  is invertible it follows that (37) is causally invertible. From [13, Th. 2], we know that we can find a stable invariant manifold for the Hamiltonian flow associated with the inverse system  $([D\Sigma_-]^T \circ \Sigma_-)^{-1}$  with output  $y_a = 0$  and its inverse Hamiltonian  $H_-^\times(x_-, p_-, y_a)$  [compare (5) and (6)]. Next, we want to factor  $[D\Sigma]^T \circ \Sigma$  as  $[D\Sigma_-]^T \circ \Sigma_-$ . In order to accomplish this, we introduce a canonical change of coordinates  $(x, p) \rightarrow (x_-, p_-)$  so that the Hamiltonian system corresponding to  $[D\Sigma]^T \circ \Sigma$ , written in terms of the new coordinates will have the form of the Hamiltonian system corresponding to  $[D\Sigma_-]^T \circ \Sigma_-$ . Furthermore,  $[D\Sigma]^T \circ \Sigma$  is causally invertible, so that we can find a Hamiltonian system for  $([D\Sigma]^T \circ \Sigma)^{-1}$  together with its inverse Hamiltonian  $H^\times(x, p, y_a)$ . In order to calculate a Lagrangian invariant manifold for  $([D\Sigma]^T \circ \Sigma)^{-1}$  we have to investigate the Hamilton–Jacobi equation given by  $H^\times(x, P_x^T(x), 0)$ , for some smooth function  $P(x)$  with  $P(0) = 0$ . The next step is too simply compare the  $H(x, p_-, u)$  with the Hamiltonian  $H_-(x_-, p_-, u)$  in the new coordinates. This leads to  $[D\Sigma_-]^T \circ \Sigma_- = [D\Sigma]^T \circ \Sigma$ .

- 2) The proof that there exists a  $\Sigma_+$  such that (36) holds is analogous to the argument above. ■

An important follow-up result to Proposition 5 is the following one that allows us to determine explicit formulas for the extremal factors  $\Sigma_-$  and  $\Sigma_+$  described previously.

**Theorem 6:** Assume that  $R$  is the nonsquare, stable, nonlinear, spectral factor of  $\Phi$  given by (28). Then, it follows that

- 1) a minimal realization of the stable, minimum phase spectral factor  $\Sigma_-$  of  $\Phi$  is given by

$$\Sigma_- : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c_1(x) + E^{-1}(x)d(x)b(x)^T P_x^-(x)^T + d(x)u \end{cases} \quad (38)$$

where  $c_1(x)$  is given by (29),  $P^-$  is the smooth solution of the Hamilton–Jacobi equation

$$\begin{aligned} & P_x^-(x) [a(x) - b(x)d(x)^T E^{-2}(x)d(x)(c_1(x) - c(x))] \\ & - \frac{1}{2}c(x)^T E^{-1}(x)d(x)b(x)^T P_x^-(x)^T \\ & - \frac{1}{2}P_x^-(x)b(x)d(x)^T E^{-1}(x)c(x) \\ & - \frac{1}{2}P_x^-(x)b(x)d(x)^T E^{-2}(x)d(x)b(x)^T P_x^-(x)^T \\ & + \frac{1}{2}c_2(x)^T c_2(x) = 0 \end{aligned} \quad (39)$$

with  $P^-(0) = 0$  and stability side condition

$$a(x) - b(x)E^{-1}(x) [d(x)^T(x)c_1(x) + b(x)^T P_x^-(x)^T] \quad (40)$$

is Lyapunov stable;

- 2) a minimal realization of the stable, maximum phase spectral factor  $\Sigma_+$  of  $\Phi$  is given by

$$\Sigma_+ : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c_1(x) + E^{-1}(x)d(x)b(x)^T P_x^+(x)^T + d(x)u \end{cases} \quad (41)$$

where  $c_1(x)$  is given by (29),  $P^+$  is the smooth solution of the Hamilton–Jacobi equation

$$\begin{aligned} P_x^+(x) [a(x) - b(x)d(x)^T E^{-2}(x)d(x) (c_1(x) - c(x))] \\ - \frac{1}{2}c(x)^T E^{-1}(x)d(x)b(x)^T P_x^+(x)^T \\ - \frac{1}{2}P_x^+(x)b(x)d(x)^T E^{-1}(x)c(x) \\ - \frac{1}{2}P_x^+(x)b(x)d(x)^T E^{-2}(x)d(x)b(x)^T P_x^+(x)^T \\ + \frac{1}{2}c_2(x)^T c_2(x) = 0 \end{aligned} \quad (42)$$

with  $P^+(0) = 0$  and antistability side condition

$$a(x) - b(x)E^{-1}(x) [d(x)^T (x)c_1(x) + b(x)^T P_x^+(x)^T] \quad (43)$$

is antistable.

*Proof:* Since  $R$ ,  $\Sigma_-$  and  $\Sigma_+$  are all spectral factors of  $\Phi$  we have that  $\Phi = [DR]^T \circ R = [D\Sigma_-]^T \circ \Sigma_- = [D\Sigma_+]^T \circ \Sigma_+$ . Hence, we may proceed via the hypothesis of Proposition 1 in the following way. From (20) and (21), it is clear that the differential equation  $\dot{x} = a(x) + b(x)u$  in the first lines of (38) and (41) will hold. In order to compute the “ $c$ ” term in the said formulas we consider (23), formula (28) and in each case choose the “ $d$ ” term to be  $d(x)$ . If, for instance, we denote the “ $c$ ” term of (38) by  $c_-(x)$ , then it follows that  $c_-(x)^T d(x) = P_x(x)b(x) + c_1(x)^T d(x)$ . If we multiply each term on the right by  $d(x)^T E^{-1}(x)$  then formula (38) will follow. Moreover, we can determine formula (41) by a similar calculation. Furthermore, the Hamilton–Jacobi equations (39) and (42) can be established by a consideration of (22) in Proposition 1. ■

The existence of the smooth solutions  $P^-$  and  $P^+$  of the Hamilton–Jacobi equations (39) and (42), respectively, are suggested by [64, Ch. 8, Cor. 8.1.18]. We note that the stable, minimum and maximum phase spectral factors described in the result above are necessarily square because of our assumption that  $E(x) = d(x)^T d(x) = d(x)d(x)^T$  is invertible. Our next task is to determine the (one-sided) right inverses of the spectral factors  $\Sigma_-$  and  $\Sigma_+$ , where we assumed that  $\Phi$  in (3) is weakly coercive. The result is as follows.

*Theorem 7:* Assume that the hypothesis of Theorem 6 holds. If we suppose that  $\Phi$  is weakly coercive then we can compute minimal realizations for the right inverses of the stable, minimum phase spectral factor  $\Sigma_-$  and the stable, maximum phase spectral factor  $\Sigma_+$  as

$$\Sigma_-^{-R} : \begin{cases} \dot{x} = a(x) - b(x)d(x)^T E^{-1}(x)c_1(x) - b(x)d(x)^T \\ \quad \times E^{-2}(x)d(x)b(x)^T P_x^-(x)^T + b(x)d(x)^T E^{-1}(x)y \\ u = -d(x)^T E^{-1}(x)c_1(x) - d(x)^T E^{-2}(x) \\ \quad \times d(x)b(x)^T P_x^-(x)^T + d(x)^T E^{-1}(x)y \end{cases} \quad (44)$$

and

$$\Sigma_+^{-R} : \begin{cases} \dot{x} = a(x) - b(x)d(x)^T E^{-1}(x)c_1(x) - b(x)d(x)^T \\ \quad \times E^{-2}(x)d(x)b(x)^T P_x^+(x)^T + b(x)d(x)^T E^{-1}(x)y \\ u = -d(x)^T E^{-1}(x)c_1(x) - d(x)^T E^{-2}(x) \\ \quad \times d(x)b(x)^T P_x^+(x)^T + d(x)^T E^{-1}(x)y \end{cases}$$

respectively.

*Proof:* We may prove the result by direct computation. ■

The following lemma provides further information about the relationships between extremal spectral factors and certain inner systems (see Section II for a characterization).

*Lemma 8:* Assume that the hypothesis of Theorem 6 holds where  $\Sigma_-$  is stable, minimum phase spectral factor of  $\Phi$ ,  $\Sigma_+$  is a stable, maximum phase spectral factor of  $\Phi$ ,  $\tilde{\Sigma}_-$  is an antistable, minimum phase spectral factor of  $\Phi$  and  $\tilde{\Sigma}_+$  is an antistable, maximum phase spectral factor of  $\Phi$ . Then there exists inner systems  $\Theta_1$ ,  $\Theta_2$ ,  $\Theta_3$  and  $\Theta_4$  such that

$$\Sigma_+ = \Theta_1 \circ \Sigma_- \quad (45)$$

$$\tilde{\Sigma}_+ = \Theta_2 \circ \Sigma_- \quad (46)$$

$$\tilde{\Sigma}_- = \Theta_3 \circ \Sigma_- \quad (47)$$

$$\Sigma_+ = \Theta_4 \circ \tilde{\Sigma}_+. \quad (48)$$

*Proof:* First, we establish that (45) holds. Since  $\Sigma_-$  and  $\Sigma_+$  are spectral factors of  $\Phi$  it follows that  $\Phi = [D\Sigma_-]^T \circ \Sigma_- = [D\Sigma_+]^T \circ \Sigma_+$ . In this case, if we are given any initial state  $x^1$  for  $\Sigma_+$ , we can choose the initial state  $(x^1, x^1)$  for  $\Theta_1 \circ \Sigma_-$  so that the input–output map for  $\Sigma_+$ , corresponds to the input–output map related to  $\Theta_1 \circ \Sigma_-$ . This argument, of course, also applies for (46) and (47). Next, we use (45) and (46) to show that (48) holds. From (45), we have that  $\Sigma_- = \Theta_1^{-1} \circ \Sigma_+$ . Replace this expression for  $\Sigma_-$  into (46) in order to obtain  $\tilde{\Sigma}_+ = \Theta_2 \circ \Theta_1^{-1} \circ \Sigma_+$ . The final step is to choose  $\Theta_4 = \Theta_1 \circ \Theta_2^{-1}$ .

It remains to show that the systems  $\Theta_1$ ,  $\Theta_2$ ,  $\Theta_3$  and  $\Theta_4$  are indeed inner. We, firstly, prove this fact for  $\Theta_1$ ; the verifications that  $\Theta_2$  and  $\Theta_3$  are inner are analogous. Since  $\Sigma_+$  and  $\Sigma_-$  in (45) are both spectral factors of  $\Phi$  we have that

$$\Phi(u) = [D\Sigma_+(u)]^T \Sigma_+(u) = [D\Sigma_-(u)]^T \Sigma_-(u).$$

In addition, we recall that  $\Sigma_-$  is an outer factor. Then the conclusion that  $\Theta_1$  in (45) is inner is an immediate consequence of [13, Th. 2] (alternatively, see [10, Prop. 3]). Because  $\Theta_4 = \Theta_1 \circ \Theta_2^{-1}$ , also  $\Theta_4$  is inner. ■

This result is a consequence of the properties of  $\Sigma$  and  $R$  as spectral factors of  $\Phi$ . The factorizations occurring in (45), (46), and (47) are all examples of *inner–outer factorizations* that have an important role to play in control theory. Moreover, we can use (41) for  $\Sigma_+$  and (44) for  $\Sigma_-^{-R}$  to compute  $\Theta_1$  appearing in (45) explicitly. In the linear case, the system  $\Theta_2$  in (46) is known as the *phase function* which, as is the case below, plays an important role in understanding minimal spectral factors. In addition, we can find linear versions of the inner system,  $\Theta_3$ , that makes (47) stable and  $\Theta_4$  that makes (48) antistable. The next result uses the outcomes of Lemma 8 to establish further connections between the extremal spectral factors.

**Theorem 9:** Assume that the hypothesis of Theorem 6 holds. If we suppose that  $\Phi$  is weakly coercive then for the stable, minimum phase spectral factor  $\Sigma_-$ , the stable, maximum phase spectral factor  $\Sigma_+$  and the antistable, maximum phase spectral factor  $\tilde{\Sigma}_+$  we have that

$$\Sigma_+ = \Sigma_+ \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+ \quad (49)$$

$$\Sigma_- = \Sigma_- \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+ \quad (50)$$

$$\Sigma_- = \Sigma_- \circ \Sigma_+^{-R} \circ \Sigma_+. \quad (51)$$

*Proof:* From  $\tilde{\Sigma}_+ \circ \tilde{\Sigma}_+^{-R} = I$ , we have  $\tilde{\Sigma}_+ \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+ = \tilde{\Sigma}_+$ . Furthermore, from (48), we see that  $\Sigma_+ \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+ = \Theta_4 \circ \tilde{\Sigma}_+ \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+ = \Theta_4 \circ \tilde{\Sigma}_+ = \Sigma_+$ . This proves (49). Next, we note that by using (45), we have

$$\Sigma_+ = \Theta_1 \circ \Sigma_- = \Sigma_+ \circ \tilde{\Sigma}_+^{-R} \circ \tilde{\Sigma}_+. \quad (52)$$

If we multiply by  $\Theta_1^{-1}$  on either side of (52) and apply (45) once again, we find that (50) holds. In order to verify the identity (51), we simply substitute  $\Sigma_+ = \Theta_1 \circ \Sigma_-$  from (45) into  $\Sigma_+ = \Sigma_+ \circ \Sigma_+^{-R} \circ \Sigma_+$ . ■

### B. Column and Row Rigid Systems

Consider the Hamiltonian system  $[D\Theta_c]^T \circ \Theta_c$  (Hamiltonian extension with  $u_a = y$ ) with Hamiltonian

$$H(x, p, u) = p^T [a(x) + b(x)u] + \frac{1}{2} [c(x) + u]^T [c(x) + u]. \quad (53)$$

Consider the observability function  $P^o$  (see [60] and the references contained therein) defined as the solution of  $P_x^o(x)a(x) + (1/2)c(x)^T c(x) = 0$  and define new coordinates  $\bar{p} = p - P_x^o(x)$ . Then

$$H(x, \bar{p}, u) = \bar{p}^T [a(x) + b(x)u] + u^T [b(x)^T P_x^o(x)^T + c(x)] + \frac{1}{2} u^T u.$$

Now, if  $P^o$  satisfies  $P_x^o(x)b(x) + c(x)^T = 0$  then the submanifold  $\bar{p}$  is an invariant manifold, and the system  $[D\Theta_c]^T \circ \Theta_c$  restricted to this manifold is given by the static input–output identity map  $u \mapsto y_a = u$ . In this case, the system  $\Theta_c$  is said to be **column rigid**. Next, consider the Hamiltonian system  $[D\Theta_r] \circ \Theta_r^T$  (Hamiltonian extension with  $u = y_a$ ) with Hamiltonian

$$H(x, p, u_a) = p^T a(x) + \frac{1}{2} p^T b(x)b(x)^T p + p^T b(x)u_a + c(x)^T u_a + \frac{1}{2} u_a^T u_a.$$

Consider the controllability function  $P^c$  (see [60] and the references contained therein) defined as the solution of  $P_x^c(x)a(x) + (1/2)P_x^c(x)c(x)c(x)^T P_x^c(x)^T = 0$  and define canonical coordinates  $\bar{p} = p - P_x^c(x)$ . Then

$$H(x, \bar{p}, u_a) = \bar{p}^T a(x) + \frac{1}{2} \bar{p}^T b(x)b(x)^T \bar{p} + P_x^c(x)^T [b(x)b(x)^T \bar{p} - b(x)u_a] + c(x)^T u_a + \frac{1}{2} u_a^T u_a.$$

Now, if  $P^c$  satisfies  $P_x^c(x)b(x) + c(x)^T = 0$  then the submanifold  $\bar{p}$  is an invariant manifold, and the system  $[D\Theta_r] \circ \Theta_r^T$  restricted to this manifold is given by the static input–output identity map  $u_a \mapsto y = u_a$ . In this case, the system  $\Theta_r$  is **row rigid**.

The next result that we obtain is one which postulates the existence of a factorization of a minimal, stable, spectral factor into a column rigid system and an outer system.

**Theorem 10:** Suppose that  $R$  is the minimal, stable, non-square, spectral factor of the weakly coercive system  $\Phi$  given by (28).

- 1) If  $\Sigma_-$  is the stable, minimum phase (outer) spectral factor of  $\Phi$  given by (38), then there exists an essentially unique column rigid system  $\Theta_c$  for which

$$R = \Theta_c \circ \Sigma_-. \quad (54)$$

- 2) If  $\Sigma_+$  is the stable, maximum phase spectral factor of  $\Phi$  given by (41), then there exists an essentially unique row rigid system  $\Theta_r$  for which

$$\Sigma_+ = \Theta_r \circ R. \quad (55)$$

*Proof:*

- 1) First, we know that if  $R$  and  $\Sigma_-$  are spectral factors of  $\Phi$ , then  $\Phi = [DR]^T \circ R = [D\Sigma_-]^T \circ \Sigma_-$ . As a result of this, we may verify that (54) is true by using an argument analogous to the one that was used to prove that (45) holds. Furthermore, we note that

$$\begin{aligned} \Phi(u) &= [DR(u)]^T \circ R(u) \\ &= [D\Sigma_-(u)]^T \circ \Sigma_-(u) \\ &= [D\Sigma_-(u)]^T \circ [D\Theta_c(u)]^T \circ \Theta_c(u) \circ \Sigma_-(u) \\ &= [D(\Theta_c \circ \Sigma_-)(u)]^T \circ (\Theta_c \circ \Sigma_-)(u) \end{aligned}$$

with  $\Sigma_-$  outer and  $[D\Theta_c(u)]^T \circ \Theta_c(u) = u$ . Then, by the previous definition,  $\Theta_c$  satisfies the criteria for a system to be column rigid.

- 2) We define  $P_r = R \circ \Sigma_+^{-R}$ , where  $\Sigma_+$  is given by (41). Also, we define  $\Theta_r = P_r^T$ , which shows that  $\Theta_r$  is rigid. Furthermore, we have to show that  $\Sigma_+ = \Theta_r \circ R$ . By using the first part with  $R = \Theta_c \circ \Sigma_-$  and  $\Sigma_- = \Sigma_- \circ \Sigma_+^{-R} \circ \Sigma_+$  from (51) of Proposition 9 we may conclude that

$$\begin{aligned} \Sigma_+ &= \Theta_r \circ P_r \circ \Sigma_+ \\ &= \Theta_r \circ [R \circ \Sigma_+^{-R}] \circ \Sigma_+ \\ &= \Theta_r \circ \Theta_c \circ \Sigma_- \circ \Sigma_+^{-R} \circ \Sigma_+ \\ &= \Theta_r \circ \Theta_c \circ \Sigma_- \\ &= \Theta_r \circ R. \end{aligned}$$

■

We can also show that Proposition 10 holds by a Morse theory argument (cf. [13, Th. 2]). The next proposition tells us that we can express the rigid systems  $\Theta_c$  and  $\Theta_r$  in terms of smooth solutions of the Hamilton–Jacobi equations (39) and (42), respectively, and components of the state–space formula for (28).

*Theorem 11:* Suppose  $\Phi = [DR]^T \circ R$  as in (3), with  $R$  given by (28).

- 1) The minimal column rigid system satisfying  $R = \Theta_c \circ \Sigma_-$  in (54) is given by

$$\Theta_c : \begin{cases} \dot{x} = a(x) - b(x)d(x)^T E^{-1}(x)c_1(x) - b(x)d(x)^T \\ \quad \times E^{-2}(x)d(x)b(x)^T P_x^-(x)^T + b(x)d(x)^T E^{-1}(x)y \\ y_a = (-E^{-1}(x)d(x)b(x)^T P_x^-(x)^T) + \begin{pmatrix} I \\ 0 \end{pmatrix} y \end{cases} \quad (56)$$

where  $P^-$  is the smooth solution of the (39) with stability side condition (40).

- 2) The minimal row rigid system satisfying  $\Sigma_+ = \Theta_r \circ R$  in (55) is given by (57), as shown at the bottom of the page, where  $P^+$  is the smooth solution of the (42) with antistability side condition (43).

*Proof:* The explicit formulas are obtained via a process of direct computation. We can compute  $\Theta_c$  in (56) directly from  $R \circ \Sigma_-^R$ , where  $R$  and  $\Sigma_-^R$  are given by (28) and (44), respectively. In order to determine the explicit formula for row rigid  $\Theta_r$  in (57) a similar approach as the one shown previously may be adopted. ■

## V. CONCLUSION AND FUTURE DIRECTIONS

This paper generalizes results on nonlinear stable square spectral factors (see [10], as well as [54] and [55]) to the less restrictive case of stable nonsquare spectral factors. In particular, we use several contributions of Ball, Helton, van der Schaft, and Petersen on the factorization of nonlinear control systems ([5], [7], [10], [13], [31], and [49]) to establish the equivalence between stable nonsquare spectral factors, invariant Lagrangian submanifolds and associated solutions of Hamilton–Jacobi inequalities. In order to accomplish this we had to consider the natural subclass of spectral factors that are stable and input affine. Here, the formulas are found to be explicit and suggestive of the linear case. Also, in all of the above, we considered the situation where  $E(x) := d(x)^T d(x)$  is invertible for each  $x$ . There are several interesting problems involving nonlinear spectral factorization (and the related inner–outer factorization) that remain open. In discussing the Hamilton–Jacobi equations arising in connection with stable, spectral factors of a given Hamiltonian system  $\Phi$  we always assumed that solutions were smooth. We expect that this smoothness assumption can be removed if one works with viscosity-sense solutions [8], [14], [18], [32], [35] of these equations. The extent to which all solutions of the Hamilton–Jacobi equation (31) are of the *a priori* special type discussed in Section III remains an interesting open question. We have not yet been able to extend our analysis to the parametrization of more general (not necessarily affine, minimal or stable) nonlinear spectral factors. However, the assumption that  $E(x) := d(x)^T d(x)$  can be removed has been discussed in [65] and further investigation is envisaged for the

outer-spectral factorization case [11]. On the other hand, great difficulty has been experienced with finding an inner-outer factorization for nonstable systems. However, it seems likely that a nonstable spectral factor can be constructed by replacing the Hamilton–Jacobi equation used in this paper by a functional Hamilton–Jacobi equation. For such a factor, we will deal with an infinite-dimensional state–space, which necessitates a reconsideration of the minimality property. In this regard, however, [16] discusses a more general nonminimal factorization with a different type of Hamilton–Jacobi equation. Also, a nonlinear version of the pole-zero cancellation technique used in [45] to solve the general inner-outer and spectral factorization problem may be an interesting but difficult prospect. Nevertheless, the questions raised in the above are likely to be useful areas for subsequent research. As for the linear case, we hope to apply the concepts developed in these papers to scenarios in stochastic realization theory, network synthesis and systems and control theory. In this regard, chemical process control is discussed in the joint paper by Ball, Petersen, and van der Schaft (cf. [11]) on noninvertible nonlinear systems. Also, it remains an open question whether our results are applicable to mechanical systems with Hamiltonian structure. Recently a study (see [49]) has been launched in order to characterize the null-pole structure of nonlinear control systems. As is well known, in the linear setting, a close relationship exists between the spectral factorization problem in systems and control theory and null-pole structure (see, for instance, [50] and [51]). An appropriate generalization of the associated “null-pole triple” approach to the nonlinear case may in all likelihood prove useful for analysing these problems. Furthermore, parametrizations of nonlinear nonsquare spectral factors in terms of coprime inner–inner factorizations is discussed in detail in [49]. The analysis in the aforementioned paper by Petersen has connections with linear geometric control, in the style of Wonham (see [68]), where output nulling controlled invariant and reachability subspaces play a natural role. It is well-known that the behavioral approach to linear systems theory that was initiated by Willems is underpinned by the analysis of rectangular (polynomial) matrices. In future, it may be possible to extend some of the recent results on optimal control by Ferrante and Zampieri (see [23]) and the work by Trentelman and Willems on dissipative systems (see [25], [26], and the references contained therein) to the nonlinear setting by considering the class of nonsquare systems discussed in the present paper. However, the notion of a nonlinear system that does not differentiate between input and output variables and makes a choice of control variables from among the systems variables is something that requires further thought. If this issue is successfully resolved, we believe that the solution of the state space  $H_\infty$ -control problem for behavioral linear systems by means of dissipativeness obtained in [24] provides ample clues for deriving a nonlinear analogue.

$$\Theta_r : \begin{cases} \dot{x} = -a(x)^T + c_1(x)^T E^{-1}(x)d(x)b(x)^T + P_x^+(x)b(x)d(x)^T E^{-2}(x)d(x)b(x)^T \\ \quad + (P_x^+(x)b(x)E^{-1}(x)d(x) - c_2(x)^T)u \\ y = -E^{-1}(x)d(x)b(x)^T + (I \ 0)u \end{cases} \quad (57)$$

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**Mark A. Petersen** was born in Elsies River, Cape Town, South Africa. He received the Ph.D. degree from the University of the Western Cape, South Africa, in 1998.

His research interests are in nonlinear control systems, matrix functions, and financial mathematics. He is currently a Professor of mathematics at the North-West University, Potchefstroom, South Africa. His short-term visits elsewhere include Virginia Polytechnic Institute and State University, Blacksburg, and the Vrije Universiteit and Centrum

voor Wiskunde en Informatica, Amsterdam, The Netherlands.

Dr. Petersen is a Member of the South African Mathematical Society (SAMS), the South African Council for Automation and Computation (SACAC), and serves on the Technical Committee for Economic Systems of the International Federation on Automatic Control (IFAC). He is the Editor-Elect of the *Proceedings of the 7th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2007)*, to be held in Pretoria, South Africa.



**Arjan J. van der Schaft** received the undergraduate and Ph.D. degrees in mathematics from the University of Groningen, Groningen, The Netherlands, in 1979 and 1983, respectively.

In 1982, he joined the Department of Applied Mathematics, University of Twente, Enschede, The Netherlands, where he is currently a Full Professor in Mathematical Systems and Control Theory. His research interests include the mathematical modeling of physical and engineering systems and the control of nonlinear and hybrid systems. He is

coauthor of the following books: *System Theoretic Descriptions of Physical Systems* (1984), *Variational and Hamiltonian Control Systems* (1987, with P.E. Crouch), *Nonlinear Dynamical Control Systems* (1990),  *$L_2$ -Gain and Passivity Techniques in Nonlinear Control* (2000), and *An Introduction to Hybrid Dynamical Systems* (2000).

Dr. van der Schaft has served as an Associate Editor for *Systems and Control Letters*, the *Journal of Nonlinear Science*, the *SIAM Journal on Control*, and the *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*. Currently, he is an Associate Editor for *Systems and Control Letters* and Editor-at-Large for the *European Journal of Control*.